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LAMINAR BOUNDARY LAYER ON A FLAT PLATE AT LOW PRANDTL NUMBER†

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Abstract—Using the method of matched asymptotic expansions, we obtain the first four terms in the low Prandtl number expansion for the recovery temperature and heat transfer in the flat plate compressible boundary layer (assuming a viscosity proportional to temperature). It is found that, provided the series are properly Eulerized, the results so obtained are good even when Prandtl number approaches infinity.

NOMENCLATURE

C, defined as $\equiv (\gamma - 1) M_{\infty}^2$;

erf(x), error function defined as

$$\equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-x^2} dx;$$

f', dimensionless velocity in x-direction;

f^*, defined by equation (11);

$G(\sigma)$, defined by equation (25a);

$K(\eta)$, defined as $\equiv f''^2(\eta)$;

M, Mach number;

N, defined as $\equiv \rho\mu/\rho_{\infty}\mu_{\infty}$;

r, recovery factor defined in (20);

s, Reynolds analogy factor defined in (26);

T, dimensionless temperature;

x, y, coordinates along the plate measured from leading edge and normal to the plate.

Greek symbols

γ, ratio of specific heats;

ζ, outer variable defined as $\equiv \sigma^{\frac{1}{2}}\eta$;

η, Howarth–Dorodnitsyn variable defined as

$$\equiv \left(\frac{\rho_{\infty}U_{\infty}}{2\mu_{\infty}x}\right)^{\frac{1}{2}} \int_0^y (\rho/\rho_{\infty})\,dy$$

μ, fluid viscosity;

ρ, fluid density;

σ, Prandtl number.

Subscripts

∞, free steam;

r, adiabatic wall;

0, wall.

Superscripts

$'$, $''$, $'''$, first, second and third derivative with respect to η.

1. INTRODUCTION

Boundary layer problems with low Prandtl number are of interest in flow of liquid metals and plasma. As we shall show here, however, a study of the limiting case of small Prandtl number σ is of value even if σ is not very small. For air the Prandtl Number is 0·73 at normal temperature and pressure, but flight at high speeds and altitudes leads to high temperatures and low pressures at which the Prandtl number can become very small (see Fig. 1 after Thomas [1]).

The general problem of compressible laminar boundary layers has been studied extensively in the past [2], but in general it is found difficult to account for arbitrary fluid parameters and

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Mach number. Except in some specific numerical calculations, accounts of which are given by several authors (notably Kuerti [3], Young [4] and Van Driest [5]), most of the results assume the viscosity of the fluid to be proportional to temperature. However, the viscosity-temperature relation is better represented either by the simple power law $\mu \propto T^{\omega}$, where ω is a constant usually between $\frac{1}{2}$ and 1, or by the Sutherland law which involves two constants and so is somewhat more complicated.

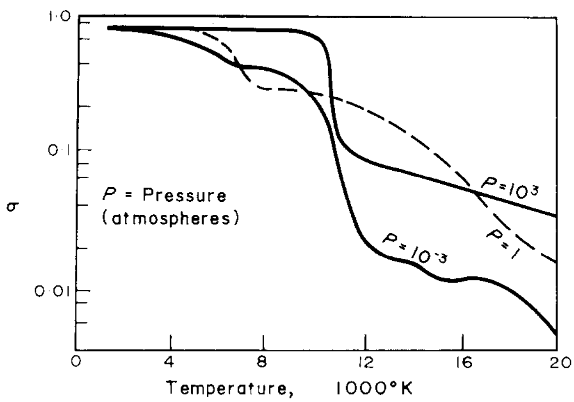

FIG. 1. Prandtl number for air, frozen mixture (after Thomas [1]).

Morgan *et al.* [6] have analysed the flow of an incompressible fluid at low σ and presented results up to order σ for the rate of heat transfer (ignoring the dissipation), and the recovery factor. Adams [7] and Goddard and Acrivos [8] have presented results up to order $\sigma^{\frac{1}{2}}$ and $\sigma^{\frac{3}{2}}$ respectively for the heat transfer (again the flow is incompressible and dissipation is ignored). Sparrow and Gregg [9, 10] have presented a summary of low σ results. Stewartson [2] gives the results up to order $\sigma^{\frac{3}{2}}$ for the recovery factor and Reynolds analogy factor for a compressible fluid with $\mu\rho$ = constant. Edward and Tellep [11] have analysed heat transfer with variable thermal properties to order $\sigma^{\frac{1}{2}}$.

In the present work, an attempt is made to improve solutions systematically for a gas with $\omega = 1$ using the method of matched asymptotic expansions. Solutions for other values of ω will be studied in a later report. In the method of matched asymptotic expansions, we generally study the inner and outer limits of the problem and try to match them in the overlap region. This enables us to construct the complete temperature profile and to obtain more accurate expressions for the recovery factor and the heat transfer at the wall. Only flows with no pressure gradient are considered in this work.

2. BASIC EQUATIONS

The boundary layer equations for steady compressible flow of a fluid past a flat plate can be written in the usual notation [2] as

$$(Nf'')' + ff'' = 0 \tag{1}$$

$$\left(\frac{N}{\sigma}T'\right)' + fT' + CNf''^2 = 0 \tag{2}$$

where $N = \rho\mu/\rho_\infty\mu_\infty$, $\quad C = (\gamma - 1)M_\infty^2$.

The dependent variables f' (proportional to the velocity) and T have been nondimensionalized with respect to free stream values. Dashes indicate differentiation with respect to the Howarth–Dorodnitsyn variable η defined by

$$\eta = \left(\frac{\rho_\infty U_\infty}{2\mu_\infty x}\right)^{\frac{1}{2}} \int_0^y (\rho/\rho_\infty)\,\mathrm{d}y.$$

The boundary conditions for the velocity profile are

$$f(0) = 0 = f'(0), f'(\infty) = 1. \tag{3a}$$

For the temperature profile we can have either an insulated plate for which

$$T'(0) = 0, \qquad T(\infty) = 1, \tag{3b}$$

or wall temperature prescribed as a constant value, say T_0; then

$$T(0) = T_0, \qquad T(\infty) = 1. \tag{3c}$$

To study the problem systematically at low Prandtl number, we consider the analysis in two parts. In this report we deal with the case when the Prandtl number and the product of density and viscosity are constants. More general variable property flows will be studied in a later report.

3. ANALYSIS

Under the assumptions stated above, the momentum equation (1) becomes explicitly independent of T. The problem reduces to one of solving the two equations

$$f''' + ff'' = 0 \tag{4}$$

$$T'' + \sigma f T' + C\sigma f''^2 = 0. \tag{5}$$

The equation (4) under boundary conditions (3a) is the well known Blasius problem, whose solution may be considered known. Its solution near the surface for small η is

$$f(\eta) = a\eta^2/2! - a^2\eta^5/5! + 11a^3\eta^8/8! - 375a^4\eta^{11}/11! + O(\eta^{14}) \tag{6}$$

where $a = 0{\cdot}469600$; and for large η is

$$f(\eta) = \eta - \beta + O\left[\frac{1}{\eta^2}\exp\{-(\eta - \beta)^2/2\}\right] = \eta - \beta + O(\eta^{-\infty}) \tag{7}$$

where $\beta = 1{\cdot}21678$, and we write $O(\eta^{-\infty})$ to denote an exponentially small term in the limit $\eta \to \infty$.

Since the Prandtl number is low, the thermal conductivity is large, so there is a small momentum layer inside a huge thermal layer. Following Lagerstrom [12] to solve (5) under boundary conditions (3b) or (3c) we proceed as follows.

3.1 *Inner solution*

First taking the inner limit, which is defined to be $\sigma \to 0$ with η fixed, we write,

$$T(\eta, \sigma) = \sum_{n=0} t^{(n)}(\eta)\sigma^{n/2}. \tag{8}$$

Substituting this series (8) into equation (5) and collecting various powers of $\sigma^{\frac{1}{2}}$ we get

$$t''^{(n)} = -ft'^{(n-2)} - Cf''^2\delta_{2n}$$

Here $K(\eta) = f''^2(\eta)$ is proportional to the viscous dissipation, δ_{ij} is the Kronecker delta and

$$t^{(-1)} = 0 = t^{(-2)}. \tag{10}$$

Let

$$f^*(\eta) = \beta - \eta + f(\eta) \tag{11}$$

where $f^*(\eta) = O(\eta^{-\infty})$ as $\eta \to \infty$.

Now with the help of (10) and (11), the solution to the first six equations in (9) can be written as

$$t^{(0)} = a_0\eta + A_0 \tag{12a}$$

$$t^{(1)} = a_1\eta + A_1 \tag{12b}$$

$$t^{(2)} = -a_0[\eta^3/6 - \beta\eta^2/2 + \int_0^\eta (\eta - \eta_1) f^*(\eta_1)\,d\eta_1] - C\int_0^\eta (\eta - \eta_1) K(\eta_1)\,d\eta_1 + a_2\eta + A_2 \tag{12c}$$

$$t^{(3)} = -a_1[\eta^3/6 - \beta\eta^2/2 + \int_0^\eta (\eta - \eta_1) f^*(\eta_1)\,d\eta_1] + a_3\eta + A_3 \tag{12d}$$

$$
\begin{aligned}
t^{(4)} = {} & C(\eta^3/6 - \beta\eta^2/2)\int_0^\eta K(\eta_1)\,\mathrm{d}\eta_1 - C\eta\int_0^\eta(\eta_1^2/2 - \beta\eta_1)K(\eta_1)\,\mathrm{d}\eta_1 \\
& + C\int_0^\eta(\eta - \eta_2)f^*(\eta_2)\int_0^{\eta_2}K(\eta_1)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 \\
& + C\int_0^\eta(\eta_1^3/3 - \beta\eta_1^2/2)K(\eta_1)\,\mathrm{d}\eta_1 \\
& + a_0[\eta^5/40 - \beta\eta^4/8 + \beta^2\eta^3/6 + (\eta^3/6 - \beta\eta^2/2)\int_0^\eta f^*(\eta_1)\,\mathrm{d}\eta_1 \\
& - \int_0^\eta(\eta_1^3/6 - \beta\eta_1^2/2)f^*(\eta_1)\,\mathrm{d}\eta_1 + \int_0^\eta(\eta - \eta_2)f^*(\eta_2)\int_0^{\eta_2}f^*(\eta_1)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2] \\
& - a_2[\eta^3/6 - \beta\eta^2/2 + \int_0^\eta(\eta - \eta_1)f^*(\eta_1)\,\mathrm{d}\eta_1] \\
& + a_4\eta + A_4
\end{aligned}
\tag{12e}
$$

$$
\begin{aligned}
t^{(5)} = {} & a_1[\eta^5/40 - \beta\eta^4/8 + \beta^2\eta^3/6 + (\eta^3/6 - \beta\eta^2/2)\int_0^\eta f^*(\eta_1)\,\mathrm{d}\eta_1 \\
& - \int_0^\eta(\eta_1^3/6 - \beta\eta_1^2/2)f^*(\eta_1)\,\mathrm{d}\eta_1 + \int_0^\eta(\eta - \eta_2)f^*(\eta_2)\int_0^{\eta_2}f^*(\eta_1)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2] \\
& - a_3[\eta^3/6 - \beta\eta^2/2 + \int_0^\eta(\eta - \eta_1)f^*(\eta_1)\,\mathrm{d}\eta_1] \\
& + a_5\eta + A_5.
\end{aligned}
\tag{12f}
$$

Here the a_i and A_i are constants of integration to be determined by boundary conditions at the wall and by matching this inner solution with the outer solution. In writing down the solutions in the particular form shown in (12), it is necessary at some places to change the order of integration suitably (this being easily justified).

It is easily seen that the solution (12) becomes singular at large η and does not satisfy the boundary condition at infinity. This singularity is rather similar to the one encountered in improving Stokes's solution in low Reynolds number flow [12]. The outer expansion of the inner solution (12) for large η is a complicated power series in η and σ and is given in Appendix A.1.

3.2 *Outer solution*

It is now clear that a different outer approximation for large η is needed. From an order of magnitude analysis we introduce the outer variable

$$\zeta = \sigma^{\frac{1}{2}}\eta \tag{13}$$

and study the limit as $\sigma \to 0$ with ζ fixed. From (7) for large η we have

$$f''(\eta) = O(\eta^{-\infty}).$$

The equation (5) with the help of (7) and (13) reduces to the outer flow equation

$$\frac{\mathrm{d}^2T}{\mathrm{d}\zeta^2} + (\zeta - \sigma^{\frac{1}{2}}\beta)\frac{\mathrm{d}T}{\mathrm{d}\zeta} = O(\sigma^\infty) \tag{14}$$

which is correct to all orders in σ, i.e. the error here is exponentially small. This is so because the thermal layer is much thicker than the momentum layer and all that the momentum layer does far away is to displace the stream lines from their inviscid position by an amount β. The outer equations of all orders in σ proceed from this simple equation which can be solved once and for all. If $z = \zeta - \sigma^{\frac{1}{2}}\beta$ equation (14) reduces to

$$\frac{\mathrm{d}^2T}{\mathrm{d}z^2} + z\,\frac{\mathrm{d}T}{\mathrm{d}z} = 0. \tag{15}$$

The solution to this equation which satisfies the boundary condition $T(\infty) = 1$ is

$$T = B + (1 - B)\,\mathrm{erf}\left(\frac{z}{\sqrt{2}}\right), \tag{16}$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt$$

is the well known error function, $B = B(\sigma)$ is a constant of integration independent of z but a function of σ: say

$$B = \sum_{n=0} B_n \sigma^{n/2}.$$

The inner expansion of the outer solution (16) as $\zeta \to 0$ is

$$T(\zeta \to 0) = \sum_{m=0} B_m \sigma^{m/2} + (2/\pi)^{\frac{1}{2}} \sum_{m=0} \frac{(-1)^m (\eta - \beta)^{2m+1}}{m!(2m+1)\,2^m}\sigma^{(2m+1)/2} - (2/\pi)^{\frac{1}{2}} \sum_{s=0}\sum_{m=0} \frac{(-1)^m (\eta - \beta)^{2m+1}}{m!(2m+1)\,2^m} \times B_s \sigma^{(2m+1+s)/2}. \tag{17}$$

3.3 *Results*

The various unknown constants in (12) and (17) will now be determined by the matching principle [13],

$$\begin{matrix}\text{Inner limit of outer} \\ \text{solution } T(\zeta \to 0)\end{matrix} = \begin{matrix}\text{Outer limit of inner} \\ \text{solution } T(\eta \to \infty).\end{matrix} \tag{18}$$

3.3(a) *Insulated plate.* Here $T'(0) = 0$ and equations (12a)–(12f) give $a_0 = a_1 = \ldots = a_5 = 0$. The matching principle (18) determines the A_i and B_i (after changing the order of integration in A_3 and A_4) as

$$\begin{aligned}
A_0 &= B_0 = 1, \\
A_1 &= B_1 = C(\pi/2)^{\frac{1}{2}} \int_0^\infty K(\eta_1)\,d\eta_1, \\
A_2 &= C \int_0^\infty (\beta - \eta_1) K(\eta_1)\,d\eta_1, \quad B_2 = 0, \\
A_3 &= B_3 = C(\pi/8)^{\frac{1}{2}} \Big[\Big\{\beta^2 - 2\int_0^\infty f^*(\eta_1)\,d\eta_1\Big\} \int_0^\infty K(\eta_2)\,d\eta_2 \\
&\quad + 2\int_0^\infty K(\eta_2) \int_0^{\eta_2} f(\eta_1)\,d\eta_1\,d\eta_2\Big], \\
A_4 &= (\beta^3/3) \int_0^\infty K(\eta_1)\,d\eta_1 - \int_0^\infty (\beta - \eta_1) f^*(\eta_1)\,d\eta_1 \int_0^\infty K(\eta_2)\,d\eta_2 \\
&\quad + \int_0^\infty K(\eta_2) \int_0^{\eta_2} (\beta - \eta_1) f(\eta_1)\,d\eta_1\,d\eta_2, \\
B_4 &= 0.
\end{aligned} \tag{19}$$

The wall recovery factor is

$$r = \frac{2(T_r - 1)}{C} = 2\sum_{n=1} A_n \sigma^{n/2}. \tag{20}$$

The integrals in (20) can be evaluated by various methods as described in Appendix A.2 to obtain

$$r = 0{\cdot}9255\,\sigma^{\frac{1}{2}} + 0{\cdot}1951\,\sigma - 0{\cdot}1661\,\sigma^{\frac{3}{2}} + 0{\cdot}0236\,\sigma^2 + 0(\sigma^{\frac{5}{2}}). \tag{21}$$

In this result (21) for the recovery factor, the first term has been given by Stewartson [2] and the first two terms by Morgan *et al.* [6].

3.3(b) *Heat transfer.* Here the wall temperature $T(0) = T_0$ is prescribed and we are interested in heat transfer. In order to satisfy the wall condition, equations (12a to 12f) require

$$A_0 = T_0, \qquad A_1 = A_2 = \ldots A_5 = 0. \tag{22}$$

The matching principle (18) determines the a_i and B_i as

$$\left.\begin{aligned}
a_0 &= 0\\
B_0 &= T_0\\
a_1 &= (2/\pi)^{\frac{1}{2}}(1 - T_0)\\
B_1 &= \beta(2/\pi)^{\frac{1}{2}}(1 - T_0)\\
a_2 &= C\int_0^\infty K(\eta_1)\,d\eta_1 - 2\beta(1 - T_0)/\pi\\
B_2 &= C\int_0^\infty \eta_1 K(\eta_1)\,d\eta_1 - 2\beta^2(1 - T_0)/\pi\\
a_3 &= (2/\pi)^{\frac{1}{2}}\left[-C\int_0^\infty \eta_1 K(\eta_1)\,d\eta_1 + (1 - T_0)\left\{\int_0^\infty f^*(\eta_1)\,d\eta_1 + \frac{4-\pi}{2\pi}\beta^2\right\}\right]\\
B_3 &= (2/\pi)^{\frac{1}{2}}\left[-\beta C\int_0^\infty \eta_1 K(\eta_1)\,d\eta_1 + (1 - T_0)\left\{\int_0^\infty \eta_1 f^*(\eta_1)\,d\eta_1 + \frac{12-\pi}{6\pi}\beta^3\right\}\right]\\
a_4 &= \frac{2\beta C}{\pi}\int_0^\infty \eta_1 K(\eta_1)\,d\eta_1 + C\int_0^\infty K(\eta_2)\int_0^{\eta_2} f(\eta_1)\,d\eta_1\,d\eta_2\\
&\quad - (1 - T_0)\left[\frac{2}{\pi}\int_0^\infty (\beta + \eta_1) f^*(\eta_1)\,d\eta_1 - \frac{4(\pi-3)}{3\pi^2}\beta^3\right]\\
B_4 &= \frac{2\beta^2 C}{\pi}\int_0^\infty \eta_1 K(\eta_1)\,d\eta_1 + C\int_0^\infty K(\eta_2)\int_0^{\eta_2} \eta_1 f(\eta_1)\,d\eta_1\,d\eta_2\\
&\quad - (1 - T_0)\left[\frac{4\beta}{\pi}\int_0^\infty \eta_1 f^*(\eta_1)\,d\eta_1 + \frac{12-\pi}{3\pi^2}\beta^4\right]\\
a_5 &= -(2/\pi)^{\frac{1}{2}}\left[C\int_0^\infty \eta_1 K(\eta_1)\,d\eta_1\int_0^\infty f^*(\eta_2)\,d\eta_2 - \frac{C\beta^2}{\pi}(4-\pi)\int_0^\infty \eta_1 K(\eta_1)\,d\eta_1\right.\\
&\quad + C\int_0^\infty K(\eta_2)\int_0^{\eta_2} \eta_1 f(\eta_1)\,d\eta_1\,d\eta_2 + (1 - T_0)\left\{-\frac{\pi+4}{2\pi}\beta\int_0^\infty \eta_1 f^*(\eta_1)\,d\eta_1\right.
\end{aligned}\right\} \tag{23}$$

$$- \frac{4-\pi}{2\pi}\beta\int_0^\infty(\beta+\eta_1)f^*(\eta_1)\,d\eta_1 - \int_0^\infty f^*(\eta_2)\int_{\eta_2}^\infty f^*(\eta_1)\,d\eta_1\,d\eta_2$$

$$- \beta^4(3\pi^2 - 40\pi + 96)/3\pi^2\Big\}\Big]$$

$$B_5 = -(2/\pi)^{\frac{1}{2}}\Big[C\beta\int_0^\infty K(\eta_2)\int_0^{\eta_2}\eta_1 f(\eta_1)\,d\eta_1\,d\eta_2 + C\beta^3\frac{12-\pi}{6\pi}\int_0^\infty \eta_1 K(\eta_1)\,d\eta_1$$

$$+ C\int_0^\infty \eta_2 K(\eta_2)\,d\eta_2\int_0^\infty \eta_1 f^*(\eta_1)\,d\eta_1$$

$$+ (1-T_0)\Big\{\int_0^\infty(\eta_1^3/6 - \beta\eta_1^2/2)\,f^*(\eta_1)\,d\eta_1 + \frac{12-\pi}{2\pi^2}\beta^2\int_0^\infty \eta_1 f^*(\eta_1)\,d\eta_1$$

$$+ \int_0^\infty \eta_2 f^*(\eta_2)\int_0^{\eta_2} f^*(\eta_1)\,d\eta_1\,d\eta_2 + \int_0^\infty \eta_2 f^*(\eta_2)\,d\eta_2\int_0^\infty f^*(\eta_1)\,d\eta_1$$

$$+ (3\pi^2 - 80\pi + 480)\,\beta^5/120\pi^2\Big\}\Big].$$

The heat-transfer rate at the wall can be expressed as

$$T'(0) = \sum_{n=0} a_n\sigma^{n/2}. \qquad (24)$$

With the help of the recovery temperature T_r as given by (20), eliminating various integrals which are multiples of C in (24), the heat-transfer rate may be written as

$$T'(0) = (T_r - T_0)\,G(\sigma) \qquad (25a)$$

where

$$G(\sigma) = (2\sigma/\pi)^{\frac{1}{2}} - 2\beta\sigma/\pi$$

$$+ (2/\pi)^{\frac{1}{2}}\Big[\int_0^\infty f^*(\eta_1)\,d\eta_1 + \frac{4-\pi}{2\pi}\beta^2\Big]\sigma^{\frac{3}{2}}$$

$$+ \Big[-\frac{2}{\pi}\int_0^\infty(\beta+\eta_1)\,f^*(\eta_1)\,d\eta_1$$

$$+ \frac{4(\pi-3)}{3\pi^2}\beta^3\Big]\sigma^2 + 0(\sigma^{\frac{5}{2}}) \qquad (25b)$$

[the quantity $G(\sigma)$ is proportional to the local Nusselt number divided by the square root of the local Reynolds number].

In this series (25b) the first term has again been given by Stewartson [2]. If the viscous dissipation is ignored, i.e. $T_r = 1$, then the first two terms in series (25b) have been given by Morgan *et al.* [6] and the first three terms by Goddard and Acrivos [8]*. The various integrals in $G(\sigma)$ are evaluated in Appendix A.2 to obtain

$$G(\sigma) = 0{\cdot}7979\,\sigma^{\frac{1}{2}} - 0{\cdot}7746\,\sigma + 1{\cdot}0321\,\sigma^{\frac{3}{2}} - 1{\cdot}3177\,\sigma^2 + 0(\sigma^{\frac{5}{2}}). \qquad (25c)$$

The Reynolds analogy factor is

$$s = \frac{T'(0)}{\sigma f''(0)\,(T_r - T_0)} = \frac{G(\sigma)}{\sigma f''(0)}$$

$$= 1{\cdot}6990\,\sigma^{-\frac{1}{2}} - 1{\cdot}6495 + 2{\cdot}1978\,\sigma^{\frac{1}{2}} - 2{\cdot}8860\,\sigma + 0(\sigma^{\frac{3}{2}}). \qquad (26)$$

The uniformly valid solution to the temperature profile, obtained by taking the union of inner and outer solutions, is

* In the numerical result (25c) for $G(\sigma)$, the first two terms are the same as those of Goddard and Acrivos [8], but in their third term there is a numerical error.

$$
\begin{aligned}
T(\eta) = {} & T_0 + C\sigma\left[\int_0^{\eta} \eta_1 K(\eta_1)\,\mathrm{d}\eta_1 + \eta \int_{\eta}^{\infty} K(\eta_1)\,\mathrm{d}\eta_1\right] \\
& + (2\sigma^3/\pi)^{\frac{1}{2}}(T_r - T_0)\left[\int_0^{\eta} \eta_1 f^*(\eta_1)\,\mathrm{d}\eta_1 + \eta\int_{\eta}^{\infty} f^*(\eta_1)\,\mathrm{d}\eta_1\right] \\
& + \sigma^2\left[C\int_0^{\eta} K(\eta_2)\int_0^{\eta_2} \eta_1 f(\eta_1)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 - C\int_0^{\eta} K(\eta_2)\,\mathrm{d}\eta_2 \int_0^{\eta} \eta_1 f^*(\eta_1)\,\mathrm{d}\eta_1\right. \\
& + C\eta\int_{\eta}^{\infty} K(\eta_2)\int_0^{\eta_2} f(\eta_1)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2 - C\eta\int_{\eta}^{\infty} K(\eta_2)\,\mathrm{d}\eta_2\int_0^{\eta} f^*(\eta_1)\,\mathrm{d}\eta_1 \\
& - C\eta\int_{\eta}^{\infty} f^*(\eta_1)\,\mathrm{d}\eta_1\int_0^{\eta} K(\eta_2)\,\mathrm{d}\eta_2 - C(\eta^3/6 - \beta\eta^2/2)\int_{\eta}^{\infty} K(\eta_1)\,\mathrm{d}\eta_1 \\
& \left. - (2\beta/\pi)(T_r - T_0)\left\{\int_0^{\eta} \eta_1 f^*(\eta_1)\,\mathrm{d}\eta_1 + \eta\int_{\eta}^{\infty} f^*(\eta_1)\,\mathrm{d}\eta_1\right\}\right] \\
& + \left[(T_r - T_0)\left\{1 - (2\sigma/\pi)^{\frac{1}{2}}\beta + 2\beta^2\sigma/\pi\right.\right. \\
& - (2\sigma^3/\pi)^{\frac{1}{2}}\left(\int_0^{\infty} \eta_1 f^*(\eta_1)\,\mathrm{d}\eta_1 + \frac{16-\pi}{6\pi}\beta^3\right) \\
& \left. + \sigma^2\left(\frac{4}{\pi}\beta\int_0^{\infty} \eta_1 f^*(\eta_1)\,\mathrm{d}\eta_1 + \frac{12-\pi}{3\pi^2}\beta^4\right)\right\} \\
& + C(\pi\sigma/2)^{\frac{1}{2}}\int_0^{\infty} K(\eta_1)\,\mathrm{d}\eta_1 \\
& + C(\pi\sigma^3/8)^{\frac{1}{2}}\left\{\left(\beta^2 - 2\int_0^{\infty} f^*(\eta_1)\,\mathrm{d}\eta_1\right)\int_0^{\infty} K(\eta_2)\,\mathrm{d}\eta_2\right. \\
& \left.\left. + 2\int_0^{\infty} K(\eta_2)\int_0^{\eta_2} f(\eta_1)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\right\}\right]\left[\operatorname{erf}\{(\sigma/2)^{\frac{1}{2}}(\eta - \beta)\}\right. \\
& \left. - \operatorname{erf}\beta(\sigma/2)^{\frac{1}{2}}\right]. \qquad (27)
\end{aligned}
$$

The uniformly valid solution for the temperature profile when the wall is insulated is obtained by the substitution $T_0 = T_r$ in (27).

4. DISCUSSION

The series (21) and (25c) obtained here for the recovery factor and heat transfer rate show good convergence for low σ [especially (21)], but as σ increases the rate of convergence becomes poor. It is well known, however, that such series do contain much information, and very useful estimates of the sum of slowly convergent or even divergent series can often be obtained by the use of various transformations which accelerate the convergence. Several interesting examples of such transformations are given by Van Dyke [13], one of the most widely used being the process called Eulerization (see also Meksyn [14]). In the present work we have used an extension of the conventional Eulerization procedure, described by Hardy [15] and called by him the (E, q) process.

The recovery factor series (21) is Eulerized in the Appendix A.3. The conventional Eulerization ($q = 1$) gives

$$\sigma^{-\frac{1}{3}} r = 0{\cdot}9255\, Y^{\frac{1}{3}} + 0{\cdot}5036\, Y^{\frac{4}{3}} + 0{\cdot}2997\, Y^{\frac{7}{3}} + 0{\cdot}0995\, Y^{\frac{10}{3}} + 0(Y^{\frac{13}{3}}) \quad (28)$$

where

$$Y = [\sigma^{\frac{1}{2}}/(1 + \sigma^{\frac{1}{2}})].$$

This leads to values within 5 per cent of the known asymptotic results as $\sigma \to \infty$. Some further improvement (to within 3 per cent of the asymptotic result) is easily achieved by a slight change in the value of the parameter q (to 0·8 from 1·0) as described in Appendix A.3. The Eulerized result (28) for the recovery factor is compared with those of various authors in Fig. 2. At $\sigma = 0{\cdot}6$ the present result (28) gives $r = 0{\cdot}7745$ compared to the exact value 0·77 [7], and the low σ asymptotic results of Stewartson [2] underestimates by about 7 per cent and that of Morgan *et al.* [6] overestimates by about 8 per cent. At $\sigma = 1$, the present result (28) gives $r = 1{\cdot}0037$ (the exact value is unity [7]). When $\sigma = 15$, our result (28) gives 3·575 and is 1 per cent above the exact value 3·54 [7] while the large σ asymptotic results of Stewartson [2] overestimates by about 34 per cent, Narasimha and Vasantha [19] underestimates by 4 per cent and that of Afzal [17]

$$r = 1{\cdot}922\,\sigma^{\frac{1}{3}} - 1{\cdot}341 + 0{\cdot}468\,\sigma^{-\frac{1}{3}} + \ldots \quad (29)$$

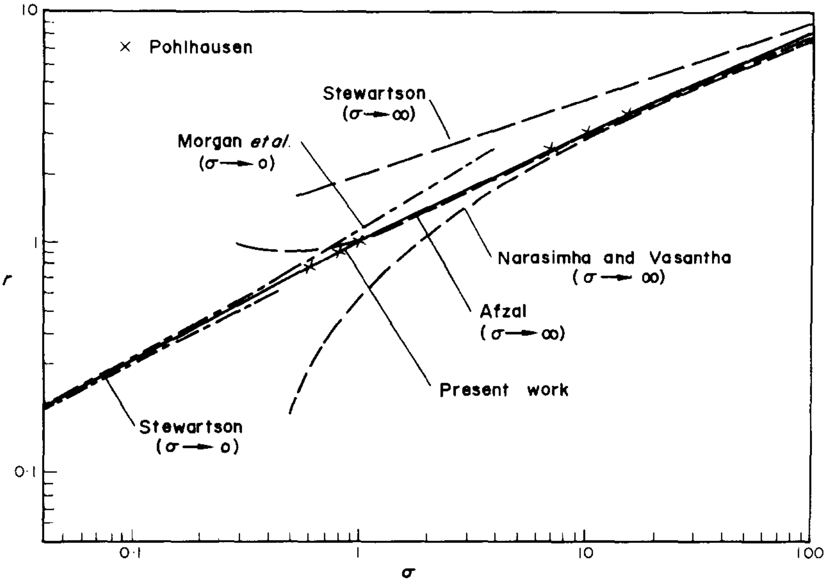

FIG. 2. Comparison of present work with previous results for recovery factor.

underestimates by 2 per cent. It should be noted that the present result (28) approaches those obtained by various workers [2, 6] as $\sigma \to \infty$, while for large σ the result (28) is close to the asymptotic result (29) and when $\sigma \to \infty$ the result (28) underestimates by less than 5 per cent.

result

$$G(\sigma) = 0{\cdot}748\,\sigma^{\frac{1}{2}}/(1 + 0{\cdot}82\,\sigma^{\frac{1}{2}})$$

of Sparrow [18] for low σ based upon von Kármán–Pohlhausen method underestimates $G(\sigma)$ throughout the range, by 6 per cent as

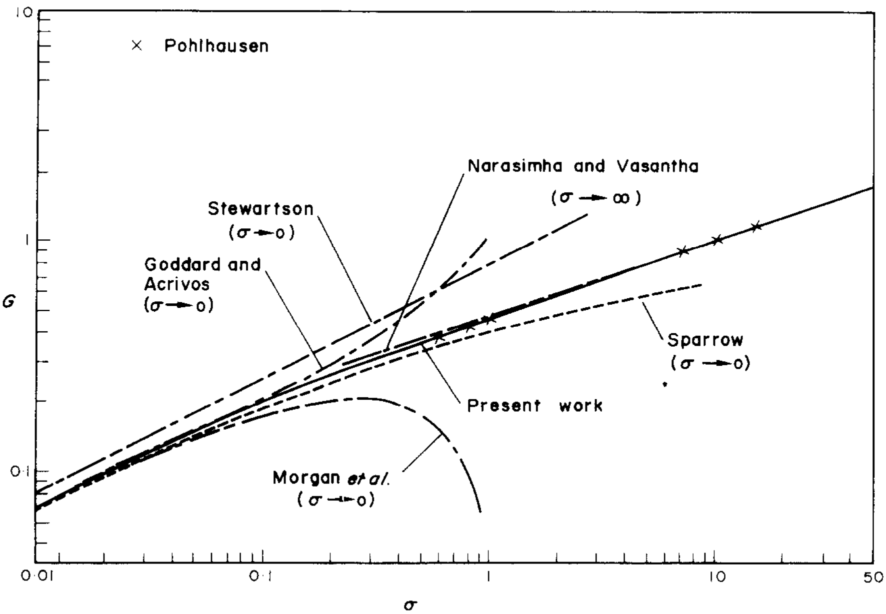

FIG. 3. Comparison of present work with previous results for the heat transfer (the results of Goddard and Acrivos [8] are plotted after correcting the numerical error in their third term).

The heat transfer series (25c) is also Eulerized in Appendix A.3 to give

$$\sigma^{-\frac{1}{3}} G = 0{\cdot}7979\, Y^{\frac{1}{3}} - 0{\cdot}5086\, Y^{\frac{4}{3}} + 0{\cdot}1766\, Y^{\frac{7}{3}} + 0{\cdot}0235\, Y^{\frac{10}{3}} + 0(Y^{\frac{13}{3}}) \quad (30)$$

and leads to values within 3 per cent of the known asymptotic result for $\sigma \to \infty$. The Eulerized result (30) is compared in Fig. 3 with that of various workers. At $\sigma = 0{\cdot}03$ the present result (30) gives $G = 0{\cdot}1194$ (the exact result [10]) and the low σ asymptotic result of Stewartson [2] overestimates by about 16 per cent, Morgan *et al.* [6] underestimates by 4 per cent and that of Goddard and Acrivos [8] (after correcting the numerical error in their third term) overestimates by 1 per cent. At $\sigma = 1$ our result gives $G = 0{\cdot}4688$ (the exact value is 0·4696 [7]). The $\sigma \to 0$ and by 12 per cent at $\sigma = 1$. It is interesting to note again that our result (30) approaches those obtained by various authors [2, 6, 8] as $\sigma \to 0$, while for large σ the result (30) is very close to the asymptotic result of Narasimha and Vasantha [19] and as $\sigma \to \infty$ the result (30) overestimates within 3 per cent.

From these results it is clear how the study of the limiting case of small Prandtl number is of use even if σ is not very small since our results can be used even for $\sigma \to \infty$ provided a sufficiently large number of terms in the expansion is computed and the series is properly Eulerized. Reliance on only the leading term in the expansion seems justified only for $\sigma < 0{\cdot}1$, and can be misleading especially in heat transfer calculations.

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APPENDIX

A.1 *Asymptotic Expansion of Inner Solution for Large η*

The outer expansion of the inner solution (12a–f) for large η is obtained (after some manipulation) as

$$
\begin{aligned}
T(\eta \to \infty) &= A_0 + a_0\eta + \sigma^{\frac{1}{2}}[A_1 + a_1\eta] \\
&+ \sigma[A_2 + a_0 \int_0^\infty \eta_1 f^*(\eta_1)\, d\eta_1 + C \int_0^\infty \eta_1 K(\eta_1)\, d\eta_1 \\
&+ \eta\{a_2 - a_0 \int_0^\infty f^*(\eta_1)\, d\eta_1 - C \int_0^\infty K(\eta_1)\, d\eta_1\} \\
&+ a_0\eta^2/2 - a_0\eta^3/6] + \sigma^{\frac{3}{2}}[A_3 + a_1 \int_0^\infty \eta_1 f^*(\eta_1)\, d\eta_1 \\
&+ \eta\{a_3 - a_1 \int_0^\infty f^*(\eta_1)\, d\eta_1\} + a_1\eta^2/2 - a_1\eta^3/6] \\
&+ \sigma^2[A_4 + a_2 \int_0^\infty \eta_1 f^*(\eta_1)\, d\eta_1 - a_0 \int_0^\infty (\eta_1^3/6 - \beta_1^2/2) \\
&\qquad \times f^*(\eta_1)\, d\eta_1 \\
&- a_0 \int_0^\infty \eta_2 f^*(\eta_2) \int_0^{\eta_2} f^*(\eta_1)\, d\eta_1\, d\eta_2 + C \int_0^\infty (\eta_1^3/3 - \beta\eta_1^2/2) \\
&\qquad \times K(\eta_1)\, d\eta_1 \\
&- C \int_0^\infty \eta_2 f^*(\eta_2) \int_0^{\eta_2} K(\eta_1)\, d\eta_1\, d\eta_2 + \eta\{a_4 - a_2 \int_0^\infty f^*(\eta_1)\, d\eta_1 \\
&+ a_0 \int_0^\infty f^*(\eta_2) \int_0^{\eta_2} f^*(\eta_1)\, d\eta_1\, d\eta_2 \\
&+ C \int_0^\infty f^*(\eta_2) \int_0^{\eta_2} K(\eta_1)\, d\eta_1\, d\eta_2 - C \int_0^\infty (\eta_1^2/2 - \beta\eta_1) \\
&\qquad \times K(\eta_1)\, d\eta_1\} \\
&+ (\beta\eta^2/2)\{a_2 - a_0 \int_0^\infty f^*(\eta_1)\, d\eta_1 - C \int_0^\infty K(\eta_1)\, d\eta_1\} \\
&+ (\eta^3/6)\{-a_2 - a_0\beta^2 + a_0 \int_0^\infty f^*(\eta_1)\, d\eta_1 + C \int_0^\infty K(\eta_1)\, d\eta_1\} \\
&- a_0\beta\eta^4/8 + a_0\eta^5/40] + \sigma^{\frac{5}{2}}[A_5 + a_3 \int_0^\infty \eta_1 f^*(\eta_1)\, d\eta_1
\end{aligned}
$$

$$- a_1 \int_0^\infty \eta_2 f^*(\eta_2) \int_0^{\eta_2} f^*(\eta_1)\, d\eta_1\, d\eta_2 - a_1 \int_0^\infty (\eta_1^3/6 - \beta\eta_1^2/2) \times f^*(\eta_1)\, d\eta_1$$

$$+ \eta\{a_5 - a_3 \int_0^\infty f^*(\eta_1)\, d\eta_1 + a_1 \int_0^\infty f^*(\eta_2) \int_0^{\eta_2} f^*(\eta_1)\, d\eta_1\, d\eta_2\}$$

$$+ (\beta\eta^2/2)\{a_3 - a_1 \int_0^\infty f^*(\eta_1)\, d\eta_1\} + (\eta^3/6)\{-a_3 - a_1\beta^2$$

$$+ a_1 \int_0^\infty f^*(\eta_1)\, d\eta_1\} - a_1\beta\eta^4/8 + a_1\eta^5/40] + O(\sigma^3).$$

A.2 *Evaluation of Integrals*

The integrals in series (21), (25) are evaluated by various methods depending upon the nature of the integral as follows:

(i) *Method of inversion of variables.* The following are the integrals which were evaluated in [20] by this method:

$$\text{(a)}\ \int_0^\infty f''^2(\eta_1)\, d\eta_1 = 0{\cdot}3692 \qquad \text{(A.2.1)}$$

$$\text{(b)}\ \int_0^\infty \eta_1 f''^2(\eta_1)\, d\eta_1 = 0{\cdot}3517 \qquad \text{(A.2.2)}$$

$$\text{(c)}\ \int_0^\infty f''^2(\eta_2) \int_0^{\eta_2} f(\eta_1)\, d\eta_1{}^2 d\eta_2 = 0{\cdot}06336 \qquad \text{(A.2.3)}$$

$$\text{(d)}\ \int_0^\infty f''^2(\eta_2) \int_0^{\eta_2} \eta_1 f(\eta_1)\, d\eta_1\, d\eta_2 = 0{\cdot}09065. \qquad \text{(A.2.4)}$$

Here $f(\eta)$ satisfies the Blasius equation (4) whose solution for small η is given by (6) and for large η by (7). The method of inversion of variables has been used by Meksyn [14] on many problems. To illustrate the procedure we consider only the last integral (d).

The first integral of the Blasius equation (4) is

$$f''(\eta) = a \exp[-F(\eta)] \qquad \text{(A.25)}$$

where $F(\eta) = \int_0^\eta f(t)\, dt$.

By inversion of variables we get, using (6),

$$\eta = \left(\frac{6F}{a}\right)^{\frac{1}{3}} \left[1 + \frac{F^2}{60} - \frac{F^2}{1260} + \frac{23F^3}{891000} + \ldots\right]. \qquad \text{(A.2.6)}$$

Now with the help of (A.2.5), (A.2.6) the integral (A.2.4) may be written as

$$(6a^5)^{\frac{1}{3}} \int_0^\infty e^{-2F} \frac{d\eta}{dF} dF \int_0^F F_1^{\frac{1}{3}} \left(1 + \frac{F_1}{60} - \frac{F_1^2}{1260} + \frac{23F_1^3}{891000} + \ldots\right) dF_1$$

$$= \left(\frac{9a^4}{16}\right)^{\frac{1}{3}} \int_0^\infty e^{-2F} F^{\frac{4}{3}} \left(1 + \frac{8}{105}F - \frac{111}{2100}F^2 + \frac{1112}{5791500}F^3 + \ldots\right) dF$$

$$= \tfrac{1}{8}(9a^4)^{\frac{1}{3}}\Gamma(\tfrac{5}{3}) \left[1 + \frac{4}{63} - \frac{11}{1890} + \frac{6116}{15637050} + \ldots\right]$$

$$= 0{\cdot}09065.$$

(ii) *Numerical integration.* The following integrals were evaluated by Simpson's rule using Smith's [21] tabulation of the Blasius solution (and same step sizes as given by Smith)

$$\text{(a)}\ \int_0^\infty f^*(t)\, dt = 1{\cdot}09131$$

$$\text{(b)}\ \int_0^\infty t f^*(t)\, dt = 0{\cdot}79699$$

$$\text{(c)}\ \int_0^\infty f^*(t) \int_0^t f^*(x)\, dx\, dt = 0{\cdot}34373$$

A.3 *Eulerization*

To improve by use of the '(E, q) method' [15] the convergence of a series

$$S = \sum_{n=0} b_n \sigma^{n\alpha + t} \qquad \text{(A.3.1)}$$

which is convergent for sufficiently small values of σ, we recast (A.3.1) in a new variable Y defined by

$$\sigma^\alpha = Y/(1 - qY), \qquad q \neq 0. \qquad \text{(A.3.2)}$$

It is easily seen that the resulting new series is

$$S = \sum_{m=0} d_m \left(\frac{\sigma^\alpha}{\sigma^\alpha + 1/q}\right)^{m + t/\alpha} \qquad \text{(A.3.3)}$$

where

$$d_m = \sum_{n=0}^{m} \binom{m + (t/\alpha) - 1}{m - n} q^{-(n + t/\alpha)} b_n.$$

For $\alpha = 1 = t$, this result (A.3.3) is due to Hardy [15].

It is well-known that the success of the usual Eulerization procedure (which puts $q = 1$ in the above series) rests on the fact that the function represented by the series (say in $\sigma^\alpha \equiv X$) possesses the nearest (to origin) singularity in the complex X-plane at $X = -1$ [13]. From (A.3.2) it appears that the (E, q) method is an extension to allow for the presence of the nearest singularity at $X = 1/q$ (by definition of radius of convergence X_c of a power series in the complex X-plane, $q = 1/X_c$). Thus if there are no other singularities the (E, q) method extends the radius of convergence to infinity.

Further, it may be noted that it can be shown [15] that if a series is summable (E, q) then for every $q' > q$, it is also summable (E, q') to the same sum. Stated in other words, if the series for S is summable (E, q) than for every

$$q' = 1/|X|, \qquad |X| < X_c \qquad \text{(A.3.4)}$$

the power series is summable (E, q') to the same sum. However, we hope that the rate of convergence should be best for $q = 1/X_c$. In practice where one usually calculates only the first few terms of a perturbation expansion, it is in general difficult to estimate the exact value of the radius of convergence and hence of q. To overcome this difficulty we can make a first guess on the circle of convergence and get q from (A.3.4) approximately. The series with this q, in general, will not give the best convergence (although the sum of the series is independent of q provided $q \geqslant 1/X_c$). Now varying the q in a neighbourhood of this value we can find by trial a q which gives the best convergence.

Let us first improve the convergence of the heat transfer series (25c). The ratios of successive coefficients in series (25c) for $G(\sigma)$ are

$$1{\cdot}030, 0{\cdot}751, 0{\cdot}783, \ldots$$

and it appears likely that these may approach unity, suggesting $q \simeq 1$. Now to recast the series (25c) in a form which hopefully will also be suitable for large σ, we extract a factor $\sigma^{\frac{1}{3}}$ and Eulerize the resulting series for $\sigma^{-\frac{1}{3}}G(\sigma)$ to yield.

$$\sigma^{-\frac{1}{3}}G = 0{\cdot}7979\left(\frac{\sigma^{\frac{1}{2}}}{1+\sigma^{\frac{1}{2}}}\right)^{\frac{1}{3}} - 0{\cdot}5086\left(\frac{\sigma^{\frac{1}{2}}}{1+\sigma^{\frac{1}{2}}}\right)^{\frac{4}{3}} \quad \text{(A.3.5)}$$

$$+ 0{\cdot}1766\left(\frac{\sigma^{\frac{1}{2}}}{1+\sigma^{\frac{1}{2}}}\right)^{7/3} + 0{\cdot}0235\left(\frac{\sigma^{\frac{1}{2}}}{1+\sigma^{\frac{1}{2}}}\right)^{10/3} + \ldots$$

As $\sigma \to \infty$ the successive partial sums of the series are

$$0{\cdot}7979, 0{\cdot}2890, 0{\cdot}4656, 0{\cdot}4891, \ldots$$

and the last partial sum is within 3 per cent (compared to the known value 0·47899 obtained from the study of the opposite limit of high σ).

Now for the recovery factor series (21), the ratios of successive coefficients are

$$4{\cdot}444, 1{\cdot}1746, 7{\cdot}038, \ldots$$

and as such it appears difficult to make a statement about the limit of sequence. However, the series (21) is more rapidly convergent than (25c), hence $q = 1$ will certainly work and the resulting series for $\sigma^{-\frac{1}{3}}r$ is

$$\sigma^{-\frac{1}{3}}r = 0{\cdot}9255\left(\frac{\sigma^{\frac{1}{2}}}{1+\sigma^{\frac{1}{2}}}\right)^{\frac{1}{3}} + 0{\cdot}5036\left(\frac{\sigma^{\frac{1}{2}}}{1+\sigma^{\frac{1}{2}}}\right)^{\frac{4}{3}}$$

$$+ 0{\cdot}2997\left(\frac{\sigma^{\frac{1}{2}}}{1+\sigma^{\frac{1}{2}}}\right)^{7/3} + 0{\cdot}0995\left(\frac{\sigma^{\frac{1}{2}}}{1+\sigma^{\frac{1}{2}}}\right)^{10/3} + \ldots . \quad \text{(A.3.6)}$$

The successive partial sums of the series as $\sigma \to \infty$ are

$$0{\cdot}9255, 1{\cdot}4291, 1{\cdot}7388, 1{\cdot}8283, \ldots$$

and the last partial sum is within 5 per cent of the known asymptotic result 1·922 as $\sigma \to \infty$.

This new series (A.3.6) cannot be expected to be the most rapidly convergent as we have used presumably too large a value of q. To select a q which gives the best convergence for large σ, we have Eulerized the series for $\sigma^{-\frac{1}{3}}r$ for various values of q and the results are as follows

q	d_0	d_1	d_2	d_3
0·5	1·1661	0·8803	0·0775	−0·7491
0·6	1·0973	0·7513	0·2108	−0·3575
0·7	1·0423	0·6613	0·2684	−0·1449
0·8	0·9970	0·5950	0·2923	−0·0217
0·9	0·9586	0·5441	0·3000	0·0529
1·0	0·9255	0·5036	0·2997	0·0995
1·1	0·8966	0·4707	0·2953	0·1291
1·2	0·8709	0·4433	0·2890	0·1481

The above results show that the best convergence is given by $q = 0{\cdot}8$ and predicts $\sigma^{-\frac{1}{3}}r$ within 3 per cent as $\sigma \to \infty$.

COUCHE LIMITE LAMINAIRE SUR UNE PLAQUE PLANE POUR UN NOMBRE DE PRANDTL FAIBLE

Résumé—Utilisant la méthode des développements asymptotiques, on obtient les quatre premiers termes dans le développement relatif au petit nombre de Prandtl pour la température de récupération et le transfert de chaleur dans la couche limite compressible sur une plaque plane (en supposant une viscosité proportionnelle à la température). On trouve que, tant que les séries sont proprement "Eulérisées", les résultats obtenus sont bons même lorsque le nombre de Prandtl approche l'infinitude.

LAMINARE GRENZSCHICHT AN EINER EBENEN PLATTE BEI KLEINEN PRANDTL-ZAHLEN

Zusammenfassung—Mit der Methode der angepassten asymptotischen Entwicklung werden die ersten vier Terme im Falle kleiner Prandtl-Zahlen für die Rückgewinn-Temperatur und den Wärmestrom in der ebenen Platte bei kompressibler Grenzschichtströmung berechnet (unter der Annahme, dass die Viskosität proportional der Temperatur ist). Es wurde fesgestellt, dass die so gefundenen Ergebnisse auch für Prandtl-Zahlen gegen Unendlich noch verwendet werden können, falls die Reihen den Euler'schen Bedingungen angepasst sind.

ЛАМИНАРНЫЙ ПОГРАНИЧНЫЙ СЛОЙ НА ПЛОСКОЙ ПЛАСТИНЕ ПРИ МАЛЫХ ЗНАЧЕНИЯХ ЧИСЛА ПРАНДТЛЯ

Аннотация—С помощью метода сращиваемых асимптотических разложений получены первые четыре члена в раздожении с малым значением числа Прандтля для температуры восстановления и переноса тепла в сжимаемом пограничном слое на плоской пластине (в предположении, что вязкость находится в пропорциональной зависимости от температуры). Найдено, что полученные таким образом результаты хороши даже, когда значение числа Прандтля стремится к бесконечности (при условии, если ряд правильно эйлеризован).